

Absolute Alpha with Limited Leverage

YIQIAO YIN

UNIVERSITY OF ROCHESTER, STUDENT

FEBRUARY, 2016

Abstract

Yin (2015) leaves an open question about leverage, l , a multiplier affecting the return of a portfolio that is not quantified. This paper proves Absolute Alpha Theory in Yin (2015) by implementing a leverage definition. We provide an economic explanation on this term by account value and portfolio value. We show consistent results with Yin (2015) and we further conclude a negative inverse relationship between marginal change on alpha w.r.t. account value and portfolio value. Lastly, we economically show the risk-return insight in portfolio management, i.e. that investors like Warren Buffett often outperform market not by winning a lot more but by not losing as much.

1 Introduction

This paper takes the model in Yin (2015) and implements a new definition for leverage, l , in the portfolio strategy. The result of the proof is consistent with the previous one in Yin (2015). Scholars as well as investment advisors have spent a lot of energy trying to invent portfolios that can achieve positive returns than market, yet no proven methods have been discovered. Instead of creating a new portfolio, one can simply invest in market with risks adjusted depending on market returns and he is able to maintain a positive alpha consistently. John Bogle discussed in his book *The Clash of the Cultures: Investment vs. Speculation* that the size of all available index funds is getting bigger in the last several decades. This model presented in Yin (2015) serves as a support so that investors can build up more conviction when they are told to invest in the market. However, Yin (2015) leaves an open question about leverage. With leverage unexplained, it is hard to understand the amount of portfolio size the strategy can implement in real world. This factor, a leverage, in Yin (2015) model is arbitrarily chosen to be bigger than one. Mathematically, it can be any number larger than one, i.e. to infinity. This is not likely the case in real life.

An investor can directly invest in market. The adjustment of risk should be proportional to price-to-moving average ratio with leverage. The intuition is as follows. An investor should be heavy on leverage when market generates non-negative returns and should be light or short on leverage when market generates negative returns. Section (2) will cover some of previous literatures on liquidity matter. Section (3) of this paper will explain the mathematics model. This paper differs from Yin (2015) in the definition of leverage, which is defined to be the portion the value of portfolio is more than the value of the account for an individual investor or an institutional investor. This section will also discuss some relationships derived from the baseline model. Section (4) will present some empirical results by using S&P 500 daily returns from 1993 to 2015 as sample.

2 Previous Literatures

Campbell and Shiller (1988) present estimates indicating that data on accounting earnings, when averaged over many years, help to predict the present value of future dividends. This result holds even when stock prices themselves are taken into consideration. The idea of forecasting power from earnings is narrowing the investment perspective to fundamental factors, which is a difficult argument to make, since fundamental factors are not the only reason driving investment decisions.

Acharya and Pedersen (2005) argued that liquidity is risky and has commonalty, which varies over time both for individual stocks and for the market as a whole. Related literatures are Chordia et al. (2000), Hasbrouck and Seppi (2001), and Huberman and Halka (1999). Their work presents a simple theoretical model, risk-averse agents in an overlapping generations economy trade securities whose liquidity varies randomly over time, explaining how asset prices are affected by liquidity risk. The model provides a unified theoretical framework that can explain the empirical findings that return sensitivity to market liquidity (Pastor and Stambaugh, 2003), that average liquidity is priced (Amihud and Mendelson, 1986), and that liquidity comoves with returns and predicts future returns (Amihud, 2002; Chordia et al., 2001; Jones, 2001; Bekaert et al., 2003).

Grossman and Miller (1988) explained market structure with two participants, market makers and outside customers. Market makers, in the secondary market, is very important for the liquidity problems.

For a security at a certain price, there are only so many shares offered for the asking price. If an investor truly believe that there is a price he would like to acquire shares for his position, he is limited to buy up to that many shares offered by the market maker at the asking price. Engle and Ferstenberg (2007) has presented the following example. A small buy order submitted as a market order will most likely execute at the asking price. If it is submitted as a limit order at a lower price, the execution will be uncertain. If it does not execute and is converted to a market order at a later time or to another limit order the ultimate price at which the order is executed will be a random variable. The variable is then thought of as having both a mean and a confidence interval. Large orders of trades usually get sent to a block desk or other intermediary who will take on the risk. They point out the relation between the risk return trade-off in this executing problem.

These literatures trigger us to push the model a step further by explaining the definition of leverage, l , since there are only so much leverage possible for an investor to take. Hence, the original model needs a condition on the multiplier l .

3 Mathematical Model

This section will explain the mathematics model and prove the absolute positive alpha under the definitions. Yin (2015) define Simple Moving Average by taking the sum of prices at any point ($P_{i,t}$) and divide the sum by the number of observations (n), namely SMA (Yin, 2013).

$$SMA(i, t)_n = \frac{1}{n} \sum_{j=1}^n P(i, t)_j \quad (1)$$

With the same concepts, we can also use exponential moving averages (EMA), defined by a weighted moving averages distributed in perspective of time between a series of prices and simple moving averages of a series of prices. The definition follows the following form.

$$EMA(i, t)_n = P_{i,t}\theta_n + SMA(i, t)_n(1 - \theta_n)$$

with $\theta = 1/(n + 1)$ to be a weight calculated exponentially. Then we have the following form,

$$EMA(i, t)_n = P_{i,t} \frac{1}{(n + 1)} + \frac{1}{n} \sum_{j=1}^n P(i, t)_j \left(1 - \frac{1}{(n + 1)}\right). \quad (2)$$

Fama (1968) mentioned Sharpe's insight of equilibrium condition and by taking derivatives he obtained the following equation.

$$E(R_i) = R_F + \left[\frac{E(R_M) - R_F}{\sigma^2(R_M)} \right] \times cov(R_i, R_M), \quad i = 1, 2, \dots, N.$$

Assuming there is a difference between access return (expected return of portfolio and risk-free return) and market premium (access return of market comparing to risk-free return). We define this difference to be alpha (α). Hence we rewrite the equation as the following.

$$r_p = r_f + \beta(r_m - r_f) + \alpha \quad (3)$$

An investor has certain amount of buying power and he can also invest heavier than his current buying power allows him to by taking a leverage. Assume he is free to add money in portfolio or liquidate positions any time he wants. He can directly invest in the market and manipulate β to be proportional to the price-to-moving average ratio or inverse price-to-moving average ratio.

$$\beta = \begin{cases} \frac{P_t}{SMA_n} = \frac{P_m}{SMA_n}, & \text{if } P_m \geq SMA_n \\ \frac{SMA_n}{P_t} = \frac{SMA_n}{P_m}, & \text{if } P_m < SMA_n \end{cases}. \quad (4)$$

Or with exponential moving averages, as the following,

$$\beta = \begin{cases} \frac{P_t}{EMA_n} = \frac{P_m}{EMA_n}, & \text{if } P_m \geq EMA_n \\ \frac{EMA_n}{P_t} = \frac{EMA_n}{P_m}, & \text{if } P_m < EMA_n \end{cases}. \quad (5)$$

Next, we can describe the return of his portfolio (r_p) to be the return of market (r_m) times β multiplied by leverage (l). We can write this relation as the following equation.

$$r_p = \begin{cases} r_m \times \beta \times l, & \text{if } r_m \geq 0 \\ r_m \times (-\beta) \times l, & \text{if } r_m < 0 \end{cases} \text{ while } l \in (1, \infty) \quad (6)$$

This relation tells us that the return of this investor's portfolio is tied up to the market with a certain leverage based on the sign of market return. When market returns positive, it makes sense for an investor to be heavy on market. The relationship between market return and portfolio return drops when market is coming down and is to cross over the moving average. This is the time when the investor should liquidate assets and invest in assets that negatively correlated to market. He can achieve this goal by buying VXX or short futures. Disregard the approaches, there is no reason for an investor to hold the market when this investment is losing money.

Under the definition of equation (4), beta is always positive and always larger than one. We define leverage to be $l = V_P/V_A - 1$ while V_P is the value of portfolio and V_A is the value of account. Leverage (l) is always larger than one as well. An investor can certain amount of money that he is willing to put in stock market. He is allowed to invest a portion of that money. He is also allowed to invest all of that money. Furthermore, he can borrow money to increase his buying power. When we say leverage, we only refer to the last scenario where an investor is borrowing money to invest. Hence, the buying power always exceeds the maximum amount of money this investor started with. That is to say, leverage (l) is always larger than one.

3.1 Absolute Alpha Theorem

Under the definitions above this investor can maintain a positive alpha by adjusting leverage and weight of his portfolio. We call this the Absolute Alpha Theory (since alpha is an absolute value, i.e. a non-negative value).

Theorem 1. *Absolute Alpha Theorem: An investor can achieve an absolute non-negative alpha by investing directly in market with beta adjusted to price-to-moving average ratio and by divesting or shorting the market with beta adjusted to negative moving average-to-price ratio.*

Proof: Assume $r_m \geq 0$, we start off by writing down Capital Asset Pricing Model. We derived the following equation from Sharpe (1964) capital asset prices model (see Appendix 6.1).

The proof (details in Appendix 6.2 and 6.3) follow the definition of beta, β , rearrange the terms, and simplify to become the form of $\alpha = \beta(l - 1)r_m$. Each terms is larger than zero for each situation, following the definition of beta, and the proof is complete.

Q.E.D.

The mathematics model also relies on the number of days used in calculating the moving averages. A moving average with less days tend to be more volatile than a moving average with more days. On the other side, a moving average with more days give you a smaller value than that with less days and hence require bigger buying power. However, bigger buying power also implies higher level of difficulty when facing less liquidity stocks. The next section will take this factor into account and evaluate the mathematics model.

The implication for this theorem is consist with Efficient Market. The theorem says an investor can generate alpha will a portfolio allocating a certain amount of risk on his portfolio. A corollary of this theorem is to say an investor is not able to unless he puts on extra risk in the portfolio.

3.2 Inverse Marginal Alpha Theorem

With the understanding Absolute Alpha Theorem, we can further discuss some minor relationship on marginal alpha in relation with value of account and value of portfolio. We introduce the following theorem, a very important implication of Efficient Market Hypothesis.

Theorem 2. *Inverse Marginal Alpha Theorem:* $\forall \alpha, \forall r_p, \exists l$ s.t. $\alpha'_p(l+1) = -\alpha'_A r_p$. Moreover, we have $-\frac{\alpha'_A}{\alpha'_P} r_p > 0$.

Proof: This proof (details in Appendix 6.4) is taking first order derivative of alpha, α , w.r.t. value of portfolio, V_P , and w.r.t. value of account, V_A . Then we can cancel out some terms by adding two equations together. After simplification of the equation, we arrive the results above.

Q.E.D.

Given an alpha, α , we can have $\alpha = \beta(l-1)r_m$. We look at the first order derivative w.r.t. value of portfolio and value of account. We can argue that the marginal rate of alpha w.r.t. value of portfolio holds inverse and negative relationship with marginal rate of alpha w.r.t. value of account. This is not a surprising statement. As a matter of fact, this statement oddly states an implication of Efficient Market Hypothesis. By the definition of leverage, we have $l+1 > 0$, so we have $-\frac{\alpha'_A}{\alpha'_P} r_p > 1 > 0$. We can then look at $-\frac{\alpha'_A}{\alpha'_P} r_p > 0$. If return of the portfolio, r_p , is positive, then either marginal rate of alpha w.r.t. portfolio value or marginal rate of alpha w.r.t. account value is positive and it follows that the other one must be negative. On the other hand, if return of the portfolio, r_p , is negative, then the sign of r_p cancels out with the negative sign at front of the equation so both marginal rates should be positive.

This insight is consistent with Efficient Market Hypothesis. You cannot achieve positive return on portfolio, positive marginal alpha w.r.t. portfolio value, and positive marginal alpha w.r.t. account value at the same time. There is a risk-return trade off in every position. For a certain amount of account size, an investor can subjectively choose a leverage at l , however, the model explained that he is not able to beat the market unless he puts on a leverage, meaning that $V_P/V_A - 1 > 0$. This action gives him an ideally positive alpha, yet at the same time increases his portfolio risks as well. If an investor has a positive portfolio return, he is either making not enough comparing to the market or making too little to increase his account value as he should have been. If an investor has a negative portfolio return, then he is probably putting himself in a risky environment when the entire market is negative. Though he may be able to beat the market in this environment, his portfolio still returns negative. We know that Warren Buffett often outperforms market not by winning a lot more but by not losing as much.

3.3 Critical Leverage Theorem

The last part of this session we introduce a lower bound of the ratio of marginal alpha w.r.t. account value and w.r.t. portfolio value.

Theorem 3. *Critical Leverage Theorem:* $\forall \alpha, \exists l^*$ s.t. $\frac{\alpha'_A}{\alpha'_P} r_p + 1 > l^*$.

Proof: This proof (see Appendix 6.5) is fairly easy. We take Theorem (2) and constrain leverage, l , to an upper bound l^* and it is complete.

Q.E.D.

The implication is fairly straightforward. For a portfolio with an alpha, we know that the ratio of marginal rate of alpha w.r.t. account value and portfolio value multiplied by return of the portfolio should be strictly less than critical leverage l^* . Critical leverage, denoted l^* , is referring to the maximum leverage allowed at a certain price level in the secondary market, which is another notation for liquidity level at that price. For a critical leverage, l^* , there must be a critical account value, V_A^* , and a critical portfolio value, V_P^* , following $l^* = V_P^*/V_A^* - 1$. Since we have $\frac{\alpha'_A}{\alpha'_P} r_p + 1 > l^*$, it follows that $\frac{\alpha'_A}{\alpha'_P} r_p > l^* - 1$. With

$l^* = V_P^*/V_A^* - 1$, it follows that $l^* - 1 = V_P^*/V_A^* - 2$. That is, an investor needs a leverage (or liquidity) level of at least 1 (or 100%) from the market maker to be big enough to have a positive ratio of marginal rate of alpha w.r.t. account value and portfolio multiplied by return of portfolio.

For an investor, he does his research and makes a decision to acquire a certain amount of shares for a position. Due to liquidity problem, he can only acquire a certain amount of shares at a certain price. If he still wants more, he will then have to be willing to pay for a higher price for extra shares. He will then move the market and increases the underlying risk in this position. For an arbitrary return of portfolio, an investor cannot really go too aggressive on the amount of risk he wants to take. If he takes too much risk, he will have to give up marginal rate of alpha. This argument is consistent with Theorem (2).

4 Data and Results

This section takes Yin (2015) results of the empirical evidence and the measurement of average daily alpha. The tests collect daily price and calculate daily return of S&P 500 from 1993 to 2015. We know that both moving average and leverage can affect the result of alpha. We chose different days of moving averages (i.e. 10-day, 20-day, 30-day, 40-day, 50-day, 100-day, 200-day, 300-day, 400-day, 500-day) in the experiment. We also chose different leverage (from 1 to 30, whole numbers).

We evaluate the model in Section (3) by calculating the average daily alpha. We have total N observations and we have total n days of moving average. We calculate the sum of all alphas and divide the sum by (N-n) to obtain the average daily alpha ($\bar{\alpha}$).

We take model by simple moving average (SMA) and calculate the average daily α with N observations when market return is non-negative:

$$\bar{\alpha}_{r_m \geq 0} = \frac{1}{N-n} \sum_{n=1}^{\infty} \alpha = \frac{1}{N-n} \sum_{n=1}^{\infty} \frac{P_m}{SMA_n} (l-1) \times r_m \quad (7)$$

We calculate the average daily α with N observations when market return is negative:

$$\bar{\alpha}_{r_m < 0} = \frac{1}{N-n} \sum_{n=1}^{\infty} \alpha = \frac{1}{N-n} \sum_{n=1}^{\infty} \frac{SMA_n}{P_m} (l-1) \times r_m \quad (8)$$

We separate equation (7) and (8) because we have different market return condition. We then calculate average daily alpha by Simple Moving Average in number of days (SMA_n) and leverage (l). We have two inputs controlled and we have the following table with average daily alpha ($\bar{\alpha}$) calculated in percentage (%). The results are consistent with Yin (2015). In Figure (1), we observe that the average daily alpha ($\bar{\alpha}$) increases when Simple Moving Average (SMA_n) and leverage (l) increase. If we control leverage ($l = 1$), then we observe that daily average alpha ($\bar{\alpha}$) increases when Simple Moving Average increases. If we control Simple Moving Average (SMA_n while $n = 10$), then we observe that daily average alpha ($\bar{\alpha}$) increases when leverage (l) increases.

From year 1993, an investor can achieve a daily average alpha ($\bar{\alpha}$) to be 0.79% if he chooses to invest all of his money ($l = 1$, i.e. not a penny more or less) into the market adjusting beta by using 10-day moving average. This would generate higher frequency of trading activities. However, he could do a lot better to increase his daily average alpha ($\bar{\alpha}$) to 0.96% ($\approx 1\%$) if he was looking at a moving average regarding more days into the past, say 500-day. He will be adjusting his portfolio less frequently than looking at less days and it will cost him less (since getting in and out of a position charges commission fee). He could also do better if he is willing to take on a leverage.

In Figure (3), we show three panels A, B, and C. Panel A and B present comparisons between portfolio return and market return. Panel C presents average return and standard deviation of average return of a portfolio under SMA_{10} with leverage l to be one. From Figure (3) Panel A we can see majority of the returns are below 0.04% for portfolio, which is consistent with results in Panel C. The results between Panel A and Panel B are consistent with the theorems in this paper. With leverage increasing, you would expect increasing average portfolio returns, but this action comes with a trade-off, i.e. you need to give up standard

deviation you could have had without leverage. With a big standard deviation, we can say that this will affect marginal rate of alpha w.r.t. portfolio value.

Figure (1). Daily prices and returns collected from 1993 to 2015. Average daily alpha ($\bar{\alpha}$) calculated in percentage (%). The row on top presents Simple Moving Average (SMA_n) with eight selected sample days (10-day, 20-day, 30-day, 40-day, 100-day, 200-day, 300-day, 400-day). The column on the left presents Leverage (l) with thirty selected sample whole numbers (1 to 5).

Ave(a) Leverage	SMA_n (in %)							
	SMA_{10}	SMA_{20}	SMA_{30}	SMA_{40}	SMA_{100}	SMA_{200}	SMA_{300}	SMA_{400}
$l=1$	0.79	0.79	0.80	0.80	0.83	0.87	0.90	0.93
$l=2$	1.61	1.62	1.63	1.64	1.69	1.76	1.83	1.90
$l=3$	2.43	2.45	2.47	2.48	2.55	2.66	2.76	2.86
$l=4$	3.25	3.28	3.30	3.32	3.41	3.56	3.69	3.82
$l=5$	4.07	4.11	4.13	4.16	4.28	4.46	4.63	4.79

Figure (2). This chart plots all the observations covering Figure (1). For each unit of Simple Moving Average (SMA_n) as x-axis, the color of the column increases from light to dark, symbolizing 10-day to 500-day. For y-axis, leverage (l) is presented from 1 to 30.

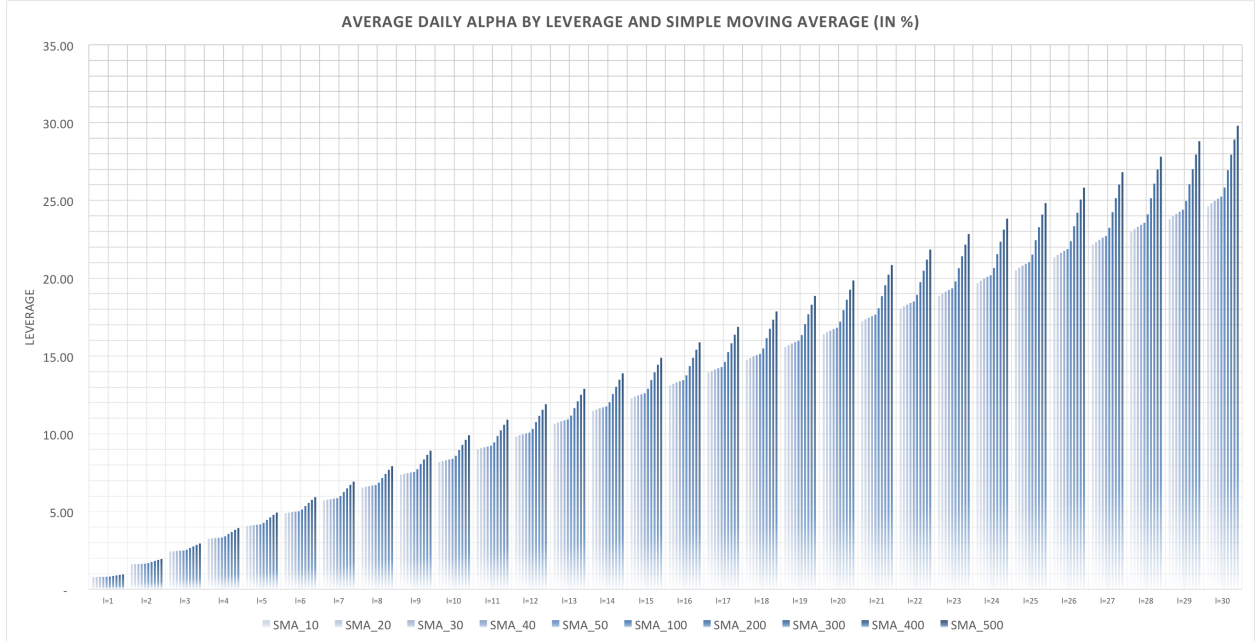


Figure (3) (Panel A). This chart plots the comparison between return of the market and the return of the portfolio given leverage of 1.

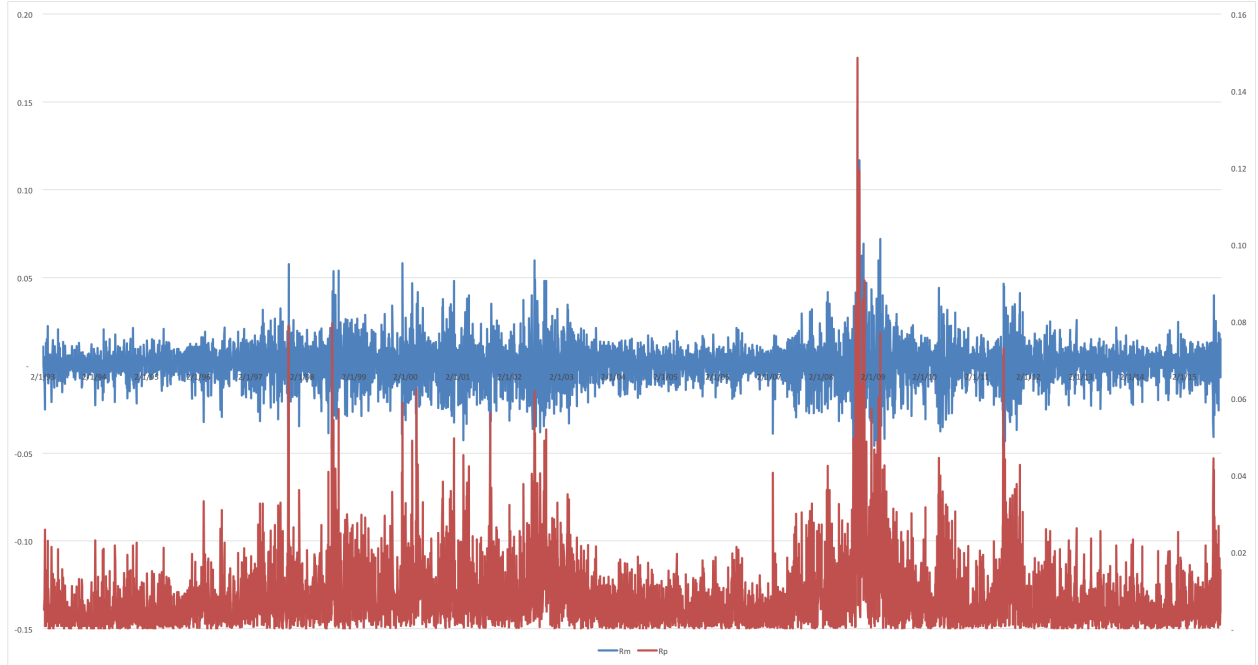


Figure (3) (Panel B). This chart plots the comparison between return of the market and the return of the portfolio given leverage of 30.

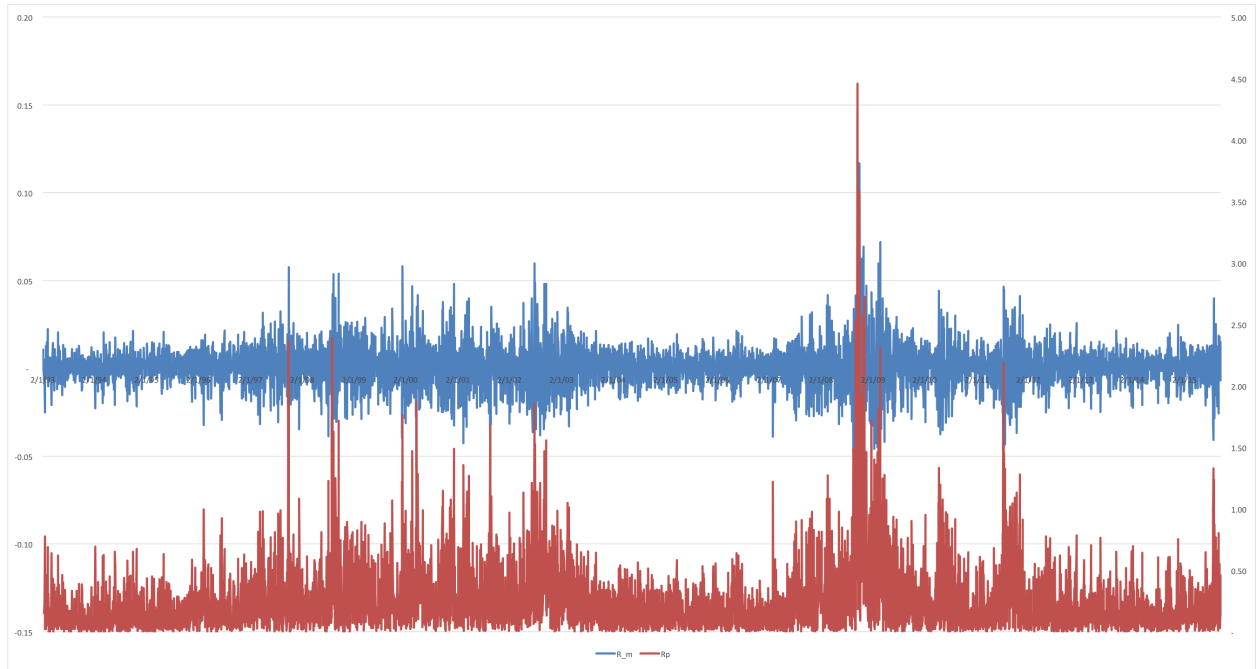


Figure (3) (Panel C). This model fixes at SMA_{10} and change leverage from $l = 1$ to $l = 5$. The table presents average returns and standard deviation.

Leverage	Average Return	Standard Deviation
1	0.008215	0.009182
2	0.016430	0.018364
3	0.024644	0.027546
4	0.032859	0.036728
5	0.041074	0.045909

5 Conclusion

This paper uses Yin (2015) as a baseline model to implement a new definition of leverage. The paper presents consistent proof to show an absolute non-negative alpha by adjusting beta with a defined leverage level. The paper further proves Inverse Marginal Alpha Theorem and Critical Leverage Theorem. Both theorems are derived from Absolute Alpha Theorem and rigorous proofs are shown in Appendix.

The grandiose intuition provided from this paper is to say that an investor can beat the market in terms of return with properly adjusting his underlying risk in portfolio. This strategy does not necessarily improve an investor's Sharpe Ratio, which leaves opening questions for future research. The paper also puts attention on one moving average in the model instead of several whereas practitioners would often rely at least multiple indicators when making investment decisions.

Rather to be a ground breaking idea, this paper implies an investor should be humble in front of market. This paper also suggests an investor to commit less frequent trading activities by selecting longer time frame of investment horizon to achieve higher alpha.

The philosophy of this paper follows value investing and Efficient Market Hypothesis. Although paper introduces an idea to beat the market, the philosophy lands on an advice to invest in market, to trade less, and to hold the portfolio in long-term.

6 Appendix

6.1 Derivation of CAPM

For a combination of two risky assets, we have the following weighted expected return, $E_{Rc} = \alpha E_{Rp} + (1 - \alpha) E_{Ra}$. We can also look at the risk by calculating the standard deviation,

$\sigma_{Rc} = \sqrt{\alpha^2 \sigma_{Rp}^2 + (1 - \alpha)^2 \sigma_{Ra}^2 + 2\alpha(1 - \alpha)\sigma_{Rp}\sigma_{Ra}}$. With all the available observations plotted on a xy-axis graph, we are interested in a combination that gives us the most optimal return-to-risk ratio. That is, we need to take derivative when one of the weight is zero ($\alpha = 0$).

We can re-arrange the standard deviation of a combination of g and i, and we have the following.

$\sigma = \sqrt{\alpha^2 \sigma_{Ri}^2 + (1 - \alpha)^2 \sigma_{Rg}^2 + 2\alpha(1 - \alpha)\sigma_{Ri}\sigma_{Rg}}$. Then we take derivate at $\alpha = 0$, and we obtain,

$$\frac{d\sigma}{d\alpha} = -\frac{1}{\sigma}[\sigma_{Rg}^2 - r_{ig}\sigma_{Ri}\sigma_{Rg}] \quad (9)$$

Next, we look at any group of point (E_{Rg}, σ_{Rg}) on the line. We are able to plug the number in equation (1). We obtain the following.

$$\frac{r_{ig}\sigma_{Ri}}{\sigma_{Rg}} = -\left[\frac{P}{E_{Rg} - P}\right] + \left[\frac{1}{E_{Rg} - P}\right]E_{Ri} \quad (10)$$

The goal here is to re-write the equation into something that we use almost everyday from equation (10). Here we are interested in the origin of capital asset pricing model? From equation (10), Sharpe (1964) defines, $B_{ig} = \frac{r_{ig}\sigma_{Ri}}{\sigma_{Rg}}$, so we can re-write equation (10). $B_{ig} = \frac{r_{ig}\sigma_{Ri}}{\sigma_{Rg}} = -\left[\frac{P}{E_{Rg} - P}\right] + \left[\frac{1}{E_{Rg} - P}\right]E_{Ri}$. We are looking at a portfolio by looking at its slope. That is, we are looking at a portfolio in terms of volatility. It gives us a very indirect picture and a vague image of what this portfolio actually looks like. We do the following derivation to make it clear.

First we want an expression with expected return on one side of the formula. From equation (2), we have:

$$B_{ig} + \frac{P}{E_{Rg} - P} = \frac{1}{E_{Rg} - P} E_{Ri}$$

Multiply both equation to get rid of the denominator on the right hand side of the formula. Then we re-write the equation with expected return on the left so that we have a mathematics expression of expected return, as described in equation (11).

$$\begin{aligned} (B_{ig} + \frac{P}{E_{Rg} - P}) \times (\frac{1}{E_{Rg} - P})^{-1} &= E_{Ri} \\ E_{Ri} &= (B_{ig} + \frac{P}{E_{Rg} - P}) \times (\frac{1}{E_{Rg} - P})^{-1} \end{aligned} \quad (11)$$

We still have a chunk of “stuff” on the right hand that we cannot visually interpret. We need to multiply these factors out of the parenthesis and re-adjust the formula into something meaningful. As we do this, we will cancel out the denominator and simply the equation, as in equation (12).

$$\begin{aligned} E_{Ri} &= B_{ig} \times (\frac{1}{E_{Rg} - P})^{-1} + \frac{P}{E_{Rg} - P} \times (\frac{1}{E_{Rg} - P})^{-1} \\ E_{Ri} &= B_{ig} \times (E_{Rg} - P) + P \end{aligned} \quad (12)$$

Based on the paper, Sharpe (1964) defined P to be pure interest rate and B to be the ratio of covariance of two risky assets and one of the risky assets. We re-name interest rate to be r_f and systematic risk from the market volatility to be β . We can re-write the equation as the following, equation (13).

$$E_{Ri} = \beta_{ig} \times (E_{Rg} - r_f) + r_f \quad (13)$$

Now we have traditional Capital Asset Pricing Model. The reason this model is powerful is because it allows investors to interpret a return of a risky asset by looking at risk-free return from treasury bill and the market access return to the risk-free return from treasury bill with systematic risk.

6.2 Proof of Absolute Alpha Theorem (part 1)

Proof: (of Absolute Alpha Theorem with SMA) Assume $r_m \geq 0$, we start off by writing down Capital Asset Pricing Model. We derived the following equation from Sharpe (1964) capital asset prices model (see Appendix 6.1).

$$r_p = r_f + \beta(r_m - r_f) + \alpha$$

Substitute r_p from equation (4) with β from equation (3) plugged into equation (4):

$$\frac{P_m}{SMA_n} l \times r_m = r_f + \beta(r_m - r_f) + \alpha$$

Rewrite equation with α on the left hand side:

$$\alpha = \frac{P_m}{SMA_n} l \times r_m - r_f - \beta(r_m - r_f) = (\frac{P_m}{SMA_n} l - \beta) \times r_m - r_f + \beta \times r_f$$

From (3), we know $\beta = \frac{P_m}{SMA_n}$. Since $r_f \approx 0$ so let $r_f = 0$, then

$$\alpha = (\frac{P_m}{SMA_n} l - \beta) \times r_m = \frac{P_m}{SMA_n} (l - 1) \times r_m \quad (14)$$

Since $r_m \geq 0$ and $l > 1$, each of the terms in equation (5) is positive. This gives us

$$\alpha \geq 0$$

Assume $r_m < 0$, we start off by writing down Capital Asset Pricing Model.

$$r_p = r_f + \beta(r_m - r_f) + \alpha$$

Substitute r_p from equation (4) with β from equation (3) plugged into equation (4):

$$-\frac{SMA_n}{P_m}l \times r_m = r_f + \beta(r_m - r_f) + \alpha$$

Rewrite equation with α on the left hand side:

$$\alpha = -\frac{SMA_n}{P_m}l \times r_m - r_f - \beta(r_m - r_f) = (-\frac{SMA_n}{P_m}l - \beta) \times r_m - r_f + \beta \times r_f$$

From (3), we know $\beta = \frac{SMA_n}{P_m}$. Since $r_f \approx 0$ so let $r_f = 0$, then

$$\alpha = (-\frac{SMA_n}{P_m}l - \beta) \times r_m = -\frac{SMA_n}{P_m}(l + 1) \times r_m \quad (15)$$

Since $r_m < 0$ and $l > 1$, negative signs cancel each other in equation (6) and this leaves the final result to be

$$\alpha > 0$$

Q.E.D.

6.3 Proof of Absolute Alpha Theorem (part 2)

Proof: (of Absolute Alpha Theorem with EMA, identical with part (1) but with EMA) Assume $r_m \geq 0$, we start off by writing down Capital Asset Pricing Model. We derived the following equation from Sharpe (1964) capital asset prices model (see Appendix 6.1).

$$r_p = r_f + \beta(r_m - r_f) + \alpha$$

Substitute r_p from equation (4) with β from equation (3) plugged into equation (4):

$$\frac{P_m}{EMA_n}l \times r_m = r_f + \beta(r_m - r_f) + \alpha$$

Rewrite equation with α on the left hand side:

$$\alpha = \frac{P_m}{EMA_n}l \times r_m - r_f - \beta(r_m - r_f) = (\frac{P_m}{EMA_n}l - \beta) \times r_m - r_f + \beta \times r_f$$

From (3), we know $\beta = \frac{P_m}{EMA_n}$. Since $r_f \approx 0$ so let $r_f = 0$, then

$$\alpha = (\frac{P_m}{EMA_n}l - \beta) \times r_m = \frac{P_m}{EMA_n}(l - 1) \times r_m \quad (16)$$

Since $r_m \geq 0$ and $l > 1$, each of the terms in equation (5) is positive. This gives us

$$\alpha \geq 0$$

Assume $r_m < 0$, we start off by writing down Capital Asset Pricing Model.

$$r_p = r_f + \beta(r_m - r_f) + \alpha$$

Substitute r_p from equation (4) with β from equation (3) plugged into equation (4):

$$-\frac{EMA_n}{P_m}l \times r_m = r_f + \beta(r_m - r_f) + \alpha$$

Rewrite equation with α on the left hand side:

$$\alpha = -\frac{EMA_n}{P_m}l \times r_m - r_f - \beta(r_m - r_f) = (-\frac{EMA_n}{P_m}l - \beta) \times r_m - r_f + \beta \times r_f$$

From (3), we know $\beta = \frac{EMA_n}{P_m}$. Since $r_f \approx 0$ so let $r_f = 0$, then

$$\alpha = (-\frac{EMA_n}{P_m}l - \beta) \times r_m = -\frac{EMA_n}{P_m}(l + 1) \times r_m \quad (17)$$

Since $r_m < 0$ and $l > 1$, negative signs cancel each other in equation (6) and this leaves the final result to be

$$\alpha > 0$$

Q.E.D.

6.4 Proof of Inverse Marginal Alpha Theorem

Proof: (of Inverse Marginal Alpha Theorem) Given $\forall \alpha, \forall r_m > 0$, and $\forall \beta$, we can choose a leverage, $l = V_P/V_A - 1$ so that marginal alpha, α' , can be computed. Recall

$$\alpha = \beta(l - 1)r_m,$$

and the definition of leverage, then we have

$$\alpha = \beta(\frac{V_P}{V_A} - 1)r_m \quad (18)$$

We take first order derivative w.r.t. value of portfolio, V_P , and w.r.t. value of account, V_A , to have the following two formulas.

$$\frac{d\alpha}{dV_P} : \alpha'_P = \frac{\beta r_m}{V_A} V'_P = \frac{\beta r_m}{V_A} r_p, \quad (19)$$

$$\text{and } \frac{d\alpha}{dV_A} : \alpha'_A = \frac{-\beta r_m V_P}{V_A^2}. \quad (20)$$

Rearrange two equations above, we get the following setting,

$$V_A \alpha'_P \frac{1}{r_p} = \beta r_m$$

$$V_A^2 \alpha'_A \frac{1}{V_P} = -\beta r_m$$

The sum of two equations above will be the following,

$$V_A \alpha'_P \frac{1}{r_p} + V_A^2 \alpha'_A \frac{1}{V_P} = 0 \quad (21)$$

Then we can multiply both sides by $1/V_A^2$ and substitute V_P/V_A by $l + 1$ to get the following

$$\alpha'_P(l + 1) + \alpha'_A r_p = 0$$

Lastly, we rearrange the formula and we arrive the theorem,

$$l + 1 = -\frac{\alpha'_A}{\alpha'_P}r_p \rightarrow (l + 1)\alpha'_P = -\alpha'_A r_p \quad (22)$$

Furthermore, we have $l + 1 > 0$, so we have

$$-\frac{\alpha'_A}{\alpha'_P}r_p > 0 \quad (23)$$

Q.E.D.

6.5 Proof of Critical Leverage Theorem

Proof: (of Critical Leverage Theorem) Given that $\forall \alpha, \exists \alpha'_A, \exists l \leq l^*$, then we take Theorem (2) $-\frac{\alpha'_A}{\alpha'_P}r_p > 0$, and subject to the constrain on leverage,

$$l^* + 1 \geq l + 1 = -\frac{\alpha'_A}{\alpha'_P}r_p$$

Since $l + 1 > 0$, we have leverage, l , strictly greater than the marginal rate of alpha w.r.t. account value and portfolio value multiplied by portfolio return. Hence, we have the following

$$\begin{aligned} l^* &\geq l > -\frac{\alpha'_A}{\alpha'_P}r_p - 1 \\ l^* &> -\frac{\alpha'_A}{\alpha'_P}r_p - 1 \\ \frac{\alpha'_A}{\alpha'_P}r_p + 1 &> l^* \end{aligned} \quad (24)$$

Q.E.D.

References

- [1] Acharya, V. and L. H. Pedersen (2005), "Asset Pricing with Liquidity Risk," *Journal of Financial Economics*, 77, 375-410.
- [2] Amihud, Y. (2002), "Illiquidity and Stock Returns: Cross-section and Time-series Effects," *Journal of Financial Markets*, 5, 31-56.
- [3] Amihud, Y., and H. Mendelson (1986), "Asset Pricing and the Bid-ask Spread," *Journal of Financial Economics*, 17, 223-249. Bekaert, G., Harvey, C.R., and C. Lundblad (2003), "Liquidity and Expected Returns: Lessons from Emerging Markets," Columbia University.
- [4] Boggle, J. (2012), "The Clash of the Cultures: Investment vs. Speculation," *Wiley & Sons, Inc.*, Hoboken, New Jersey.
- [5] Campbell, J. Y., R. J. Shiller (1988), "Stock Prices, Earnings, and Expected Dividends," *The Journal of Finance*, 43, 28-30.
- [6] Chordia, T., R. Roll, and A. Subrahmanyam (2000), "Commonality in Liquidity," *Journal of Financial Economics*, 56, 3-28.
- [7] Chordia, T., R. Roll, and A. Subrahmanyam (2001), "Market Liquidity and Trading Activity," *Journal of Finance*, 56, 501-530.

- [8] Engle, R. and R. Ferstenberg (2007), "Execution Risk," *Journal of Portfolio Management*, 33, 34-45.
- [9] Fama E.F. (1968), "Risk, Return and Equilibrium: Some Clarifying Comments," *The Journal of Finance*, 23, 29-40.
- [10] Grossman, S., and M. Miller (1988), "Liquidity and Market Structure," *Journal of Finance*, 43, 617-633.
- [11] Hasbrouck, J., and D. J. Seppi (2001), "Common Factors in Prices, Order Flows and Liquidity," *Journal of Financial Economics*, 59, 383-411.
- [12] Huberman, G., and D. Halka (1999), "Systematic Liquidity," Columbia Business School.
- [13] Jones, C.M. (2001), "A Century of Stock Market Liquidity and Trading Costs," Graduate School of Business, Columbia University.
- [14] Sharpe, W. F. (1964), "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk," *The Journal of Finance*, 19, 425-442.
- [15] Yin, Y. (2013), "How to understand Future Returns of a Security?" *Journal of Undergraduate Research*, 12, 34-39.
- [16] Yin, Y. (2015), "Absolute Alpha by Beta Manipulation," Available at SSRN: <http://ssrn.com/abstract=2733940>
- [17] Yin, Y. (2016), "Empirical Study on Greed," Available at SSRN: <http://ssrn.com/abstract=2734500>